Newton's 2nd Law

Just like $\Sigma \vec{F} = m\vec{a}$, we can say

$$\Sigma \vec{\tau} = I\vec{\alpha}$$

The big difference is you need to actually draw the forces acting on the object to figure out the torques. (Free Body Diagrams don't help.)

Energy

* The biggest change is that there is one more type of mechanical energy - the kinetic energy of rotation

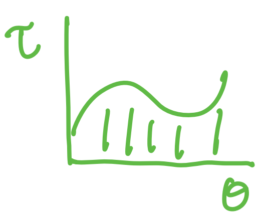
$$K = \frac{1}{2} I \omega^2$$

This means you have to ask yourself a 4th question when figuring out energy "Is anything rotating?"

* Something that does come up a lot with rotation is compound objects rotating. (e.g. a mass on the end of a rod.) When dealing with potential energy, you can treat each part of the object separately (how far each mass falls) or you can figure out the center of mass of the compound object and figure out that height and use the total mass.

* Just like $W = \int F dx$ $$W = \int \tau \, d\theta$$

Notice how the units work: Nm·rad - but radians are dimensionless, so now Nm really does become energy and we could also say J.



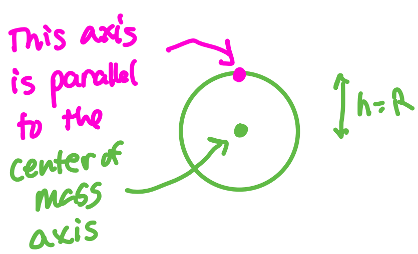
Area = Work

Parallel Axis Theorem

Because of symmetry, it is pretty straight forward to determine the moment of inertia about the center of mass of a uniform object - but objects are often rotating about a different axis. This is where the Parallel Axis Theorem comes in:

$$I = I_{cm} + mh^2$$

If the moment of inertia about the center of mass is I_{cm}, the moment of inertia about any axis PARALLEL to that is given by the equation, where m is the mass of the object and h is the distance between the two parallel axes.



For example, imagine we have a disc of mass M and radius R.

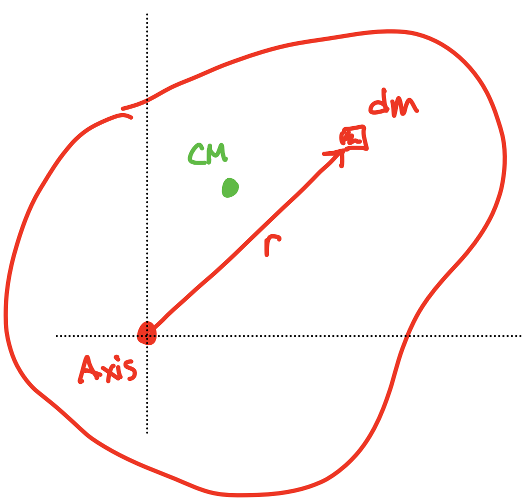
$$I_{cm} = \frac{1}{2}MR^2$$

The moment of inertia about an axis touching the edge of disc (pink in the diagram) would therefore be

$$I = I_{cm} + mh^2$$
$$= \frac{1}{2}MR^2 + MR^2$$

$$\boxed{I = \frac{3}{2}MR^2}$$

DERIVATION

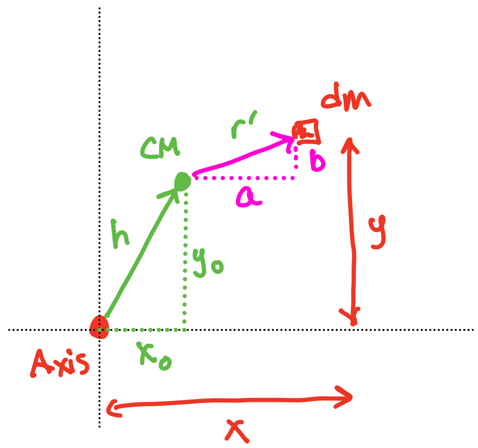


Let's determine the moment of inertia of an object about an axis as shown. The green dot is the center of mass.

By definition

$$I = \int r^2 dm$$

A random chunk "dm" is shown and the distance between "dm" and the axis is "r"



(x,y) are the coordinates of the little mass

(x_0, y_0) are the coordinates of the center of mass

(a,b) are the coordinates of the little mass with respect to the center of mass.

That means $x = x_0 + a$ and $y = y_0 + b$

Since $r^2 = x^2 + y^2$

$$r^2 = (x_0+a)^2 + (y_0+b)^2$$

$$= x_0^2 + 2ax_0 + a^2 + y_0^2 + 2y_0 b + b^2$$

So $I = \int r^2 dm$

$$= \int \left[x_0^2 + 2ax_0 + a^2 + y_0^2 + 2y_0 b + b^2\right] dm$$

$$= \int (x_0^2 + y_0^2)\, dm \rightarrow = \int h^2 dm = h^2 \int dm = Mh^2$$

$$+ \int (a^2 + b^2)\, dm \rightarrow = I_{cm} \text{ by definition}$$

$$+ \int 2ax_0\, dm \rightarrow = 2x_0 \int a\, dm = 0! \quad *$$

$$+ \int 2by_0\, dm \rightarrow = 2y_0 \int b\, dm = 0! \quad *$$

* Remember that the x coordinate of the center of mass is given by $x_{cm} = \frac{1}{M} \int x\, dm$

Since the variable a is the x coordinate of "dm" with respect to the center of mass, the integral has to be equal to 0 by definition. Likewise, $\int b\, dm$ is 0 because b is measured from the center of mass.

$$dU = (dm) g y \qquad y: 0 \to L$$

$$\therefore \quad U = \int g y \, dm$$

$$dm = \frac{M}{L} dy$$

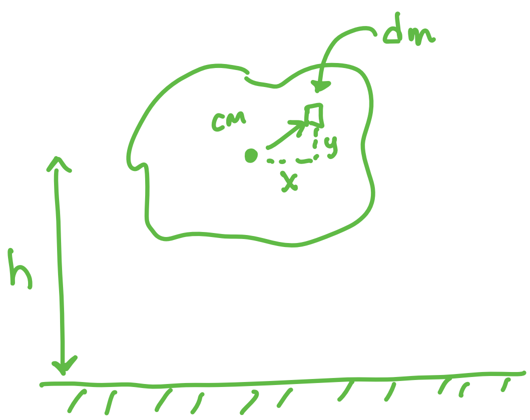
$$\text{So} \quad U = \int_0^L g y \left(\frac{M}{L}\right) dy$$

$$= \frac{Mg}{L} \int_0^L y \, dy$$

$$= \frac{Mg}{L} \left. \frac{y^2}{2} \right|_0^L = \frac{1}{2} Mg \frac{L^2}{L}$$

$$= \frac{1}{2} MgL$$

$$= Mg \left(\frac{L}{2}\right)$$



$$dU = (dm) g (h+y)$$

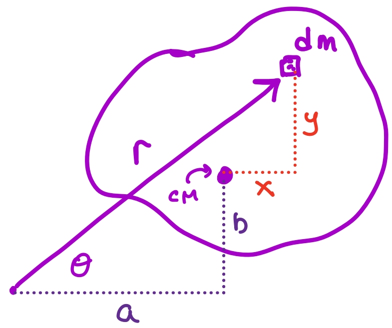
$$U = \int g(h+y) \, dm$$

$$= g \int (h+y) \, dm$$

$$= g \int h \, dm + g \int y \, dm$$

$$= gh \int dm + 0$$

$= mgh$



$$r = \sqrt{(a+x)^2 + (b+y)^2}$$

$$\cos\theta = \frac{a+x}{r}$$

$$d\vec{\tau} = r \times F = \vec{r} \times (dm)\vec{g}$$

$$d\tau = r\,(dm)\,g\cos\theta$$

$$\tau = \int d\tau$$

$$= \int r(dm)g\left(\frac{a+x}{r}\right)$$

$$= \int g(a+x)\,dm$$

$$\tau = \int ga\,dm + \underbrace{\int gx\,dm}_{=0 \text{ b/c center of mass}}$$

$$= ga\int dm$$

$$\tau = Mga \rightarrow a = r_{cm}\cos\theta$$

$$\text{so } \vec{\tau} = \vec{r}_{cm} \times m\vec{g}$$